

TREES AND GAPS FROM A CONSTRUCTION SCHEME

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ABSTRACT. We present simple constructions of trees and gaps using a general construction scheme that can be useful in constructing many other structures. As a result, we solve a natural problem about Hausdorff gaps in the quotient algebra $\mathcal{P}(\omega)/\text{Fin}$ found in the literature. As it is well known Hausdorff gaps can sometimes be filled in ω_1 -preserving forcing extensions. There are two natural conditions on Hausdorff gaps, dubbed S and T in the literature, that guarantee the existence of such forcing extensions. In part, these conditions are motivated by analogies between fillable Hausdorff gaps and Souslin trees. While the condition S is equivalent to the existence of ω_1 -preserving forcing extensions that fill the gap, we show here that its natural strengthening T is in fact strictly stronger.

1. INTRODUCTION

Souslin trees are important set-theoretic objects that were first considered in connection with the Souslin Hypothesis that characterizes the unit interval as the unique ordered continuum satisfying the countable chain condition (see [6]). They are also important tools in many other considerations in set theory. Similarly, Hausdorff's (ω_1, ω_1) -gaps in the quotient algebra $\mathcal{P}(\omega)/\text{Fin}$ are important set theoretic tools that naturally show up in a wide range considerations in set theory and related areas (see, for example, [3]). It turns out that there are numerous analogies between (ω_1, ω_1) -gaps and Aronszajn trees, trees of height ω_1 that have countable levels and branches (see, for example, [7]). Souslin trees are very specific kind of Aronszajn trees since they may admit uncountable branches in ω_1 -preserving forcing extensions of the set-theoretic universe. Analogously, as it is well known, some (ω_1, ω_1) -gaps may be filled in ω_1 -forcing extensions of the universe, so this sort of gaps are sometimes called *Souslin gaps*, or in short S-gaps. A Ramsey-theoretic analysis of S-gaps further strengthens this analogy and more importantly points out to a natural variation of this notion, a notion of a T-gap that we introduce below. We show that these seemingly similar notions are in fact different. For showing this we introduce a technique that should find many other applications. In [9] the second author introduced the concept of a construction scheme to build examples of compact spaces, convex sets and normed spaces which had previously required forcing constructions (see [1] and [8]). The existence of construction schemes is deduced in [9] from Jensen's \diamond -principle. The following are our specific results where the notions of 'capturing construction scheme', 'S-gap', 'fillable gap' and 'T-gap' are defined in the following section.

Theorem 1.1. *Assume there is a Construction Scheme that is 3-capturing. Then there is a Souslin tree.*

2010 *Mathematics Subject Classification.* 03E05, 03E35, 03E65.

Key words and phrases. Construction schemes, Suslin tree, destructible gaps, S-gaps, T-gaps.

Theorem 1.2. *Assume there is a 3-capturing Construction Scheme. Then there is a (ω_1, ω_1) -gap that is a T-gap and so, in particular there is (ω_1, ω_1) -gap that can be filled in a forcing extension over a partially ordered set satisfying the countable chain condition.*

Every T-gap is a fillable gap but the converse need not be true. More precisely, we have the following result

Theorem 1.3. *There is a model of set theory in which there is a fillable (ω_1, ω_1) -gap but with no T-gaps.*

2. PRELIMINARIES

For bounded subsets $A, B \subset \omega_1$ we say that $A < B$ if for every $a \in A$ and $b \in B$ we have $a < b$. We will work with a special kind of Δ -systems.

Definition 2.1. For $\gamma \leq \omega_1$, we say that a sequence $(s_\alpha)_{\alpha < \gamma}$ of finite subsets of ω_1 is an increasing Δ -system with root s if for every $\alpha < \beta < \gamma$ we have $s_\alpha \cap s_\beta = s$ and $s < (s_\alpha \setminus s) < (s_\beta \setminus s)$.

Recall that every uncountable family of finite subsets of ω_1 contains an uncountable increasing Δ -system as above.

2.1. Construction schemes. In this section, we introduce the notion of a construction scheme. The key feature of this scheme is that it provides a family \mathcal{F} of finite subsets of ω_1 which allow us to perform recursive constructions by amalgamating *many isomorphic* structures of lower rank. These amalgamations will determine the behaviour of uncountable substructures of the limit structure via an appropriate property of *capturing* of the construction scheme. For a more detailed analysis, see [9].

Definition 2.2. Let $(m_k)_{k < \omega}$, $(n_k)_{1 \leq k < \omega}$ and $(r_k)_{1 \leq k < \omega}$ be sequences of natural numbers such that $m_0 = 1$, $m_{k-1} > r_k$ for all $k > 0$, $n_k > k$ and for every $r < \omega$ there are infinitely many k 's with $r_k = r$. If for every $k > 0$ we have

$$m_k = n_k(m_{k-1} - r_k) + r_k$$

we say that $(m_k, n_k, r_k)_{k < \omega}$ forms a *type*.

Definition 2.3. Let \mathcal{F} be a family of finite subsets of ω_1 such that

- (1) For every $A \subset \omega_1$ finite, there is $F \in \mathcal{F}$ such that $A \subset F$.

We say that \mathcal{F} is a *construction scheme of type* $(m_k, n_k, r_k)_{k < \omega}$ if there are two mappings

$$\rho : \mathcal{F} \longrightarrow \omega \quad R : \mathcal{F} \longrightarrow [\omega_1]^{<\omega}$$

such that for every $F \in \mathcal{F}$, with $\rho^F = k > 0$ the following holds

- (2) $|F| = m_k$ and $|R(F)| = r_k$.
(3) there are unique $F_i \in \mathcal{F}$ ($i < n_k$) such that, $\rho^{F_i} = k - 1$ and

$$F = \bigcup_{i < n_k} F_i$$

Furthermore $(F_i)_{i < n_k}$ forms an increasing Δ -system with root $R(F)$, i.e.,

$$R(F) < F_0 \setminus R(F) < \dots < F_{n_k-1} \setminus R(F)$$

We call ρ^F the *rank* of F and the sequence $(F_i)_{i < n_k}$ of (3) the *canonical decomposition* of F .

It is proved in [9] that for any type $(m_k, n_k, r_k)_{k < \omega}$ there is a construction scheme with that type. To avoid confusion we will use m_k, n_k and r_k as above and we will omit reference to the type of a construction scheme.

For two $F, E \in \mathcal{F}$ of the same rank there is a unique order-preserving bijection, we denote this map by $\varphi_{F,E}$. In the particular case of $\varphi_{F_0, F_i} : F_0 \rightarrow F_i$ we will simply write φ_i when there is no confusion. If f is a function on F_0 then we can define the function $\varphi_i(f)$ in F_i by $\gamma \mapsto f(\varphi_i^{-1}(\gamma))$.

We introduce now the concept of capturing

Definition 2.4. Let \mathcal{F} be a construction scheme. We say that \mathcal{F} is *n-capturing* if for every uncountable Δ -system $(s_\xi)_{\xi < \omega_1}$ of finite subsets of ω_1 with root s there is a sub- Δ -system $(s_{\xi_i})_{i < n}$ and $F \in \mathcal{F}$ such that

$$\begin{aligned} s &\subset R(F) \\ s_0 \setminus s &\subset F_0 \setminus R(F) \\ \varphi_i(s_{\xi_0} \setminus s) &= s_{\xi_i} \setminus s \subset F_i \setminus R(F) \quad (i < n), \end{aligned}$$

where $F = \bigcup_{i < n_k} F_i$ is the canonical decomposition of F with $k = \rho^F > 0$.

In [9] it is shown that the existence of a Construction Scheme which is k -capturing for arbitrarily long $k < \omega$, follows from \diamond and can be used to construct a large spectrum of different examples of mathematical structures motivated by some previous forcing constructions (see, [1] and [8]). We will see below that only 3-capturing is enough to construct some other interesting combinatorial objects.

2.2. Gaps in $[\omega]^\omega$. We recall the definition of gap in $[\omega]^\omega$ as well as some well known results.

For a and a' , infinite subsets of ω we say $a \subseteq^* a'$ if $a \setminus a'$ is finite.

Definition 2.5. We say $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$, with $a_\alpha, b_\alpha \subset \omega$ infinite, is a *pre-gap* if for every $\alpha < \beta < \omega_1$

- (1) $a_\alpha \cap b_\alpha = \emptyset$.
- (2) $a_\alpha \subseteq^* a_\beta$ and $b_\alpha \subseteq^* b_\beta$.
- (3) $a_\delta \cap b_\gamma$ is finite for every $\delta, \gamma < \omega_1$.

We say that $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a *gap* if it is a pre-gap and

- (4) there is no infinite $c \subset \omega$ such that
 - (a) $a_\alpha \subseteq^* c$ for every $\alpha < \omega_1$.
 - (b) $b_\alpha \cap c$ is finite for every $\alpha < \omega_1$.

The existence of gaps is due to Hausdorff [5]. Recall the following Ramsey property of gaps

Proposition 2.6. A pre-gap $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ form an (ω_1, ω_1) -gap if and only if for every uncountable $\Gamma \subset \omega_1$ there are $\alpha < \beta$ in Γ such that $a_\alpha \cap b_\beta \neq \emptyset$.

Definition 2.7. We say a gap $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a *S-gap* if for every uncountable $\Gamma \subset \omega_1$ there are $\alpha < \beta$ in Γ such that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$.

The existence of a S-gap is independent of ZFC. In [4] a S-gap is constructed using \diamond and the next proposition implies that under MA_{\aleph_1} all gaps are indestructible (see e.g. [10]).

Proposition 2.8. *The following are equivalent:*

- (1) *There is an ω_1 -preserving forcing notion that splits $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$.*
- (2) *The forcing notion defined by $p \in \mathbb{P} = [\omega_1]^{<\omega}$ iff $a_\alpha \cap b_\beta = \emptyset$ for all $\alpha \neq \beta$ in p ordered by extension has the ccc.*
- (3) *For every uncountable $\Gamma \subset \omega_1$ there are $\alpha < \beta$ in Γ such that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$.*

In the literature, (ω_1, ω_1) -gaps with these properties are called ‘destructible gaps’, ‘fillable gaps’, ‘Souslin gaps’ or ‘S-gaps.’ This definition leads us to the following natural strengthening.

Definition 2.9. We say a gap $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a *T-gap* if for every uncountable $\Gamma \subset \omega_1$ there are $\alpha < \beta$ such that $a_\alpha \subseteq a_\beta$ and $b_\alpha \subseteq b_\beta$.

We will show that it is consistent that there are S-gaps but no T-gaps.

We give the proofs of the propositions for the convenience of the reader.

Proof of Proposition 2.6. Suppose $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is not a gap and let $c \subset \omega$ witness this. There is $n < \omega$ and uncountable $\Gamma \subset \omega_1$ such that $a_\alpha \setminus c \subset n$ and $b_\alpha \cap c \subset n$ for all $\alpha \in \Gamma$. We can also assume that there are $s, t \subset n$ such that for every $\alpha \in \Gamma$ $a_\alpha \cap n = s$ and $b_\alpha \cap n = t$. The condition $a_\alpha \cap b_\alpha = \emptyset$ implies that $s \cap t = \emptyset$.

For every $\alpha < \beta$ in Γ we have

$$a_\alpha \cap b_\beta = (a_\alpha \cap n) \cap (b_\beta \cap n) = s \cap t = \emptyset$$

Suppose now that there is $\Gamma \subset \omega_1$ uncountable such that $a_\alpha \cap b_\beta = \emptyset$ for every $\alpha < \beta$ in Γ . Define

$$c = \bigcup_{\alpha \in \Gamma} a_\alpha$$

is clear that $a_\alpha \subset^* c$ for every $\alpha < \omega_1$. We just have to check that $c \cap b_\gamma$ is finite for all $\gamma < \omega_1$. Let $\gamma < \omega_1$. Since $a_\alpha \cap b_\gamma$ is finite, if $c \cap b_\gamma$ is infinite there must be some $\delta \in \Gamma$ limit in Γ , $\gamma < \delta$ such that

$$\bigcup_{\alpha \in \Gamma \cap \delta} a_\alpha \cap b_\gamma \text{ is infinite}$$

but $b_\gamma \setminus b_\delta$ is finite and $\bigcup_{\alpha \in \Gamma \cap \delta} a_\alpha \cap b_\delta = \emptyset$, contradiction. □

Proof of Proposition 2.8. First we see $(3) \Rightarrow (2) \Rightarrow (1)$. Let \mathbb{P} be as in (2). Notice that \mathbb{P} forces $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ to split by forcing $\Gamma \subset \omega_1$ without the property of Proposition 2.6. We see that \mathbb{P} is ccc hence ω_1 -preserving.

Let $(p_\alpha)_{\alpha < \omega_1}$ in \mathbb{P} . There is uncountable $\Gamma \subset \omega_1$ such that:

- (i) $(p_\alpha)_{\alpha \in \Gamma}$ forms a Δ -system with $|p_\alpha| = k$.
- (ii) If $p_\alpha = \{\delta_1^\alpha < \dots < \delta_k^\alpha\}$ there is $n < \omega$ such that $a_{\delta_i^\alpha} \setminus n \subset a_{\delta_k^\alpha}$ and the same for $b_{\delta_i^\alpha}$.
- (iii) There are $s_i, t_i \subset n$ for $i = 1, \dots, k$ such that $a_{\delta_i^\alpha} \cap n = s_i$ and $b_{\delta_i^\alpha} \cap n = t_i$.

Note that $s_i \cap t_j = \emptyset$. Consider $\{\delta_k^\alpha\}_{\alpha \in \Gamma}$ by hypothesis

$$\text{there are } \alpha < \beta \text{ in } \Gamma \text{ such that } (a_{\delta_k^\alpha} \cap b_{\delta_k^\beta}) \cup (a_{\delta_k^\beta} \cap b_{\delta_k^\alpha}) = \emptyset.$$

By (iii) we have $(a_{\delta_i^\alpha} \cap b_{\delta_j^\beta}) \cup (a_{\delta_j^\beta} \cap b_{\delta_i^\alpha}) \cap n = \emptyset$, by (ii) we have

$$(a_{\delta_i^\alpha} \cap b_{\delta_j^\beta}) \cup (a_{\delta_j^\beta} \cap b_{\delta_i^\alpha}) \setminus n \subset (a_{\delta_k^\alpha} \cap b_{\delta_k^\beta}) \cup (a_{\delta_k^\beta} \cap b_{\delta_k^\alpha}) = \emptyset$$

and $p_\alpha \cup p_\beta \in \mathbb{P}$.

(2) \Rightarrow (3) Let Γ be an uncountable subset of ω_1 . Take $(p_\alpha = \{\alpha\})_{\alpha \in \Gamma}$ since \mathbb{P} has the ccc there is $\alpha < \beta$ in Γ such that $p_\alpha \not\leq p_\beta$ but this implies $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$ as we wanted.

(1) \Rightarrow (2) Let \mathbb{Q} be a forcing notion ω_1 -preserving that splits $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$. By the proof of Proposition 2.6 for every $\dot{\Gamma}_0 \subset \omega_1$ uncountable we can find $\dot{\Gamma}$ such that

$$\mathbb{Q} \Vdash \dot{\Gamma} \subset \dot{\Gamma}_0 \text{ uncountable.}$$

$$\mathbb{Q} \Vdash \text{“for every } \alpha < \beta \text{ in } \dot{\Gamma}, (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset\text{.”}$$

Applying (2) \Leftrightarrow (3), which we already proved, $\mathbb{Q} \Vdash \text{“}\mathbb{P} \text{ has the ccc”}$. If \mathbb{P} has an uncountable antichain on the ground model it has an uncountable antichain on $V^\mathbb{Q}$ because \mathbb{Q} is ω_1 -preserving. Thus \mathbb{P} is ccc and we finish the proof. \square

3. SUSLIN TREE

In this section we will prove Theorem 1.1. Let \mathcal{F} be a construction scheme 3-capturing. We will construct by recursion *finite approximations* to an uncountable tree using the structure of \mathcal{F} , then the capturing property of \mathcal{F} will make this tree Suslin.

More precisely; for every $F \in \mathcal{F}$ and every $\alpha \in F$ we construct functions $f_\alpha^F, g_\alpha^F : F \rightarrow \{0, 1\}$ such that

- (1) $f_\alpha^F \upharpoonright \alpha = g_\alpha^F \upharpoonright \alpha$
- (2) $f_\alpha^F(\alpha) = 0, g_\alpha^F(\alpha) = 1$.

We want the functions to be isomorphic and coherent;

- (3) If $E, F \in \mathcal{F}$ with $\rho^E = \rho^F$, $\alpha \in E$ and $\bar{\alpha} = \varphi_{E,F}(\alpha)$ then, $f_{\bar{\alpha}}^F = \varphi_{E,F}(f_\alpha^E)$.
- (4) If $E \subset F$, then for every $\alpha \in E$ we have

$$f_\alpha^E \subseteq f_\alpha^F \quad \text{and} \quad g_\alpha^E \subset g_\alpha^F$$

We can now define $(h_\alpha : \alpha < \omega)$ such that

$$h_\alpha \upharpoonright F = f_\alpha^F \upharpoonright (\alpha \cap F) = g_\alpha^F \upharpoonright (\alpha \cap F)$$

for every $F \in \mathcal{F}$ with $\alpha \in F$. Note that $h_\alpha : \alpha \rightarrow \{0, 1\}$ and is well defined by (3) above and property (1) of Definition 2.3. Now let

$$\mathbb{S} = (h_\alpha \upharpoonright \delta : \delta \leq \alpha < \omega_1) \tag{3.1}$$

Proof of Theorem 1.1. The functions $f_\alpha^F, g_\alpha^F (\alpha < \omega_1)$ will be defined by recursion based on the rank of $F \in \mathcal{F}$.

For $\rho^F = 0$ we have $F = \{\alpha\}$ and we let $f_\alpha^F(\alpha) = 0$ and $g_\alpha^F(\alpha) = 1$.

Let $F \in \mathcal{F}$ with $\rho^F > 0$, $R(F) = R$. Let $F = \bigcup_{i < n} F_i$ be its canonical decomposition and for all $i < n$, $f_\alpha^{F_i}, g_\alpha^{F_i}$ are defined for all $\alpha \in F_i$ satisfying (1)–(4).

Let $\varphi_i : F_0 \rightarrow F_i$ be the increasing bijection between F_0 and F_i .

For $\alpha \in R$ let $f_\alpha^F = \bigcup_{i < n} \varphi_i(f_\alpha^{F_0})$ and $g_\alpha^F = \bigcup_{i < n} \varphi_i(g_\alpha^{F_0})$.

For $\delta \in F_{2i} \setminus R$ and $\delta = \varphi_{2i}(\alpha)$ for some $\alpha \in F_0$ let

$$\begin{aligned} f_\delta^F &= \bigcup_{j \leq 2i} \varphi_j(f_\alpha^{F_0}) \cup \bigcup_{j > 2i} \varphi_j(g_\alpha^{F_0}) \\ g_\delta^F &= \bigcup_{j < 2i} \varphi_j(f_\alpha^{F_0}) \cup \bigcup_{j \geq 2i} \varphi_j(g_\alpha^{F_0}) \end{aligned}$$

For $\delta \in F_{2i+1} \setminus R$ and $\delta = \varphi_{2i+1}(\alpha)$ for some $\alpha \in F_0$ let

$$\begin{aligned} f_\delta^F &= \bigcup_{j < 2i} \varphi_j(g_\alpha^{F_0}) \cup \bigcup_{j \geq 2i} \varphi_j(f_\alpha^{F_0}) \\ g_\delta^F &= \bigcup_{j \leq 2i} \varphi_j(g_\alpha^{F_0}) \cup \bigcup_{j > 2i} \varphi_j(f_\alpha^{F_0}) \end{aligned}$$

It follows that for every $i < n$ and every $\alpha \in F_i$, $f_\delta^{F_i} \subset f_\delta^F$ and $g_\alpha^{F_i} \subset g_\alpha^F$ and (1)–(4) are preserved. This finish the recursion.

Let $\mathbb{S} \subset 2^{<\omega_1}$ be as in (3.1).

Claim 3.1. *If \mathcal{F} is a 3-capturing construction scheme, then \mathbb{S} is a Suslin tree.*

Proof. It is clear that \mathbb{S} has height ω_1 since for every $\alpha < \omega_1$, $h_\alpha \in \mathbb{S}$. Next we see that \mathbb{S} has neither uncountable antichains not uncountable chains.

Let $W = (h_\alpha \restriction \delta_\alpha : \delta_\alpha \leq \alpha, \alpha \in \Gamma) \subset \mathbb{S}$ with $\Gamma \subset \omega_1$ uncountable.

There are $\alpha < \beta$ in Γ and $F \in \mathcal{F}$ such that F captures α and β . In particular $\beta = \varphi_1(\alpha)$ and then $h_\alpha \subset h_\beta$ which implies $(h_\alpha \restriction \delta_\alpha) \not\perp (h_\beta \restriction \delta_\beta)$. This implies \mathbb{S} has no uncountable antichains.

In particular, the levels of \mathbb{S} are countable and we can find an uncountable $\Gamma_0 \subset \Gamma$ such that for every $\alpha < \beta$ in Γ_0 , $\alpha < \delta_\beta$. Let $F \in \mathcal{F}$, 3-capture Γ_0 . Thus there are $\alpha_0 < \alpha_1 < \alpha_2$ in Γ_0 captured by $F = \bigcup_{i < n_k} F_i$. By the construction we have that $h_{\alpha_1}(\alpha_0) = g_{\alpha_0}^{F_0}(\alpha_0) = 1$ and $h_{\alpha_2}(\alpha_0) = f_{\alpha_0}^{F_0}(\alpha_0) = 0$ and since $\alpha_0 < \delta_{\alpha_1}, \delta_{\alpha_2}$ then $h_{\alpha_1} \perp h_{\alpha_2}$. Thus \mathbb{S} does not have uncountable chains. \square

We showed that \mathbb{S} is a Suslin tree which is what we wanted. \square

4. T-GAP

We construct a T-gap by recursively building finite approximations $(a_\alpha^F, b_\alpha^F : \alpha \in F)$ and $(N_k)_{k < \omega}$ such that

- (1) For $\rho^F = k$ and every $\alpha \in F$, $a_\alpha^F, b_\alpha^F \subset N_k$ and $a_\alpha^F \cap b_\alpha^F = \emptyset$.
- (2) For $E, F \in \mathcal{F}$, $\rho^E = \rho^F$. If $\alpha \in E$ and $\bar{\alpha} = \varphi_{E,F}(\alpha)$ then

$$\begin{aligned} a_\alpha^E &= a_{\bar{\alpha}}^F \\ b_\alpha^E &= b_{\bar{\alpha}}^F \end{aligned}$$

- (3) If $E \subset F$ with $\rho^E = l < \rho^F$, then
 - (a) For every $\alpha \in E$, $a_\alpha^F \cap N_l = a_\alpha^E$ and $b_\alpha^F \cap N_l = b_\alpha^E$.
 - (b) For every $\alpha < \beta$ in E , $a_\alpha^F \setminus N_l \subset a_\beta^F$ and $b_\alpha^F \setminus N_l \subset b_\beta^F$.
 - (c) For every $\alpha, \beta \in E$, $a_\alpha^F \cap b_\beta^F \subset N_l$.

Proof of Theorem 1.2. For $F = \{\alpha\}$ let $a_\alpha^F = \{0\}$ and $b_\alpha^F = \{1\}$ and $N_0 = 2$.

Suppose that $(a_\alpha^E, b_\alpha^E : \alpha \in E, \rho^E < k)$ satisfies (1)–(3). For $F \in \mathcal{F}$, $\rho^F = k$, if

$$F = \bigcup_{i < n} F_i \text{ is the canonical decomposition of } F.$$

For $\alpha \in R(F)$ let $a_\alpha^F = a_\alpha^{F_0}$ and $b_\alpha^F = b_\alpha^{F_0}$.

For $\delta \in F_{2i} \setminus R(F)$ and $\delta = \varphi_{2i}(\alpha)$ for some $\alpha \in F_0$ let

$$\begin{aligned} a_\delta^F &= a_\alpha^{F_0} \cup \{N_{k-1}\} \\ b_\delta^F &= b_\alpha^{F_0} \cup \{N_{k-1} + 1\} \end{aligned}$$

For $\delta \in F_{2i+1} \setminus R(F)$ and $\delta = \varphi_{2i+1}(\alpha)$ for some $\alpha \in F_0$ let

$$\begin{aligned} a_\delta^F &= a_\alpha^{F_0} \cup \{N_{k-1} + 1\} \\ b_\delta^F &= b_\alpha^{F_0} \cup \{N_{k-1}\} \end{aligned}$$

Let $N_k = N_{k-1} + 2$. It is clear that $a_\delta^F \cap b_\delta^F = \emptyset$ and (1)–(3) are satisfied. This finish the recursion

For $\alpha < \omega_1$ let

$$a_\alpha = \bigcup \{a_\alpha^F : \alpha \in F \in \mathcal{F}\} \quad b_\alpha = \bigcup \{b_\alpha^F : \alpha \in F \in \mathcal{F}\}$$

Conditions (1)–(3) imply that $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a pre-gap.

We use Proposition 2.6 and Definition 2.9 to see that $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a T-gap. Let $\Gamma \subset \omega_1$ uncountable. Since \mathcal{F} is 3-capturing there is $F \in \mathcal{F}$ of rank k and $\alpha_0 < \alpha_1 < \alpha_2$ in Γ captured by F i.e., $\alpha_i \in F_i \setminus R(F)$ for $i < 3$ and $\alpha_j = \varphi_j(\alpha_0)$ for $j = 1, 2$. By the construction of $a_{\alpha_i}, b_{\alpha_i}$ ($i < 3$) we have that $a_{\alpha_i} \cap N_k = a_{\alpha_i}^F$ and $b_{\alpha_i} \cap N_k = a_{\alpha_i}^F$. This and (b) of (3) give

$$a_{\alpha_0} \cap b_{\alpha_1} \neq \emptyset \tag{4.1}$$

$$a_{\alpha_0} \subset a_{\alpha_2} \quad \text{and} \quad b_{\alpha_0} \subset b_{\alpha_2} \tag{4.2}$$

Equation (4.1) implies $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a gap and by (4.2) it is a T-gap as we wanted to see. \square

5. T-GAPS VS S-GAPS

We prove Theorem 1.3.

Theorem 5.1. *There is a model in which there is an S-gap but which does not have any T-gaps.*

Proof. We start with a ground model in which GCH holds and has an S-gap.

Let $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ be a gap with the property that $a_\beta \not\subset a_\alpha$ for any $\alpha < \beta < \omega_1$. It is clear that every gap is equivalent to a gap with this property. Let $\mathbf{A} = (a_\alpha)_{\alpha < \omega_1}$ and consider the following forcing notion

$$\mathbb{P}_{\mathbf{A}} = \{p \in [\mathbf{A}]^{<\omega} : (\forall x \neq y \in p) \ x \not\subset y \text{ and } y \not\subset x\}$$

ordered by reversed inclusion.

Claim 5.2. $\mathbb{P}_{\mathbf{A}}$ is ccc.

Proof. Let $(p_\alpha)_{\alpha < \omega_1}$. Applying the Δ -system Lemma we can assume that the p_α 's are a disjoint with $|p_\alpha| = n$ and $p_\alpha = (x_{\alpha,i})_{i < n}$ for every $\alpha < \omega_1$ where we preserved the natural order in \mathbf{A} . This implies that $x_{\beta,j} \not\leq x_{\alpha,i}$ for $\alpha < \beta$ and $i, j < n$.

Let M be a countable elementary submodel of H_{\aleph^+} and $\gamma = \omega_1 \cap M$.

Take $\beta > \gamma$ and fix $k < \omega$ such that

$$x_{\beta,i} \cap k \not\leq x_{\gamma,i} \quad \forall i < n. \quad (5.1)$$

Consider $\Gamma = \{\alpha < \omega_1 : x_{\alpha,i} \cap k = x_{\beta,i} \cap k \quad \forall i < n\}$, then $\Gamma \in M$ and $\beta \in \Gamma$. Therefore Γ is uncountable. Take $\alpha \in M \cap \Gamma$, by (5.1)

$$x_{\alpha,i} \not\leq x_{\gamma,i} \quad \forall i < n$$

and $p_\alpha \cup p_\gamma \in \mathbb{P}_{\mathbf{A}}$ witness $p_\alpha \not\leq p_\gamma$. \square

We will force a model where MA_{ω_1} holds for a forcing of the form $\mathbb{P}_{\mathbf{A}}$. First, fix a bijective mapping $\pi : \omega_2 \rightarrow \omega_2 \times \omega_2$ where $\pi(\alpha) = (\beta, \gamma)$ with $\beta \leq \alpha$. This is the usual book keeping mapping. Suppose we have $\mathbb{P}_\lambda = \langle \mathbb{P}_\alpha, \dot{\mathbf{Q}}_\alpha : \alpha < \lambda \rangle$ a finite support iteration with

$$\mathbb{P}_\alpha \Vdash \text{“}\dot{\mathbf{Q}}_\alpha = \mathbb{P}_{\mathbf{A}} \text{ if } \dot{\mathbf{A}} \text{ is a gap”}.$$

for some $\dot{\mathbf{A}} \in V^{\mathbb{P}_\alpha}$. Then, in $V^{\mathbb{P}_\lambda}$ there are \aleph_2 many names for gaps (by GCH), and we can fix a well-ordering of them. If $\pi(\lambda) = (\beta, \gamma)$, let $\dot{\mathbf{A}}$ be the γ^{th} name for a gap in $V^{\mathbb{P}_\beta}$. If $\dot{\mathbf{A}}$ is a gap in $V^{\mathbb{P}_\lambda}$ then let $\dot{\mathbf{Q}}_\lambda = \mathbb{P}_{\dot{\mathbf{A}}}$.

Claim 5.3. The finite support iteration \mathbb{P}_{ω_2} is ccc and forces MA_{ω_1} for orderings of the form $\mathbb{P}_{\mathbf{A}}$.

Proof. Let \mathbf{A} and $\vec{\mathcal{D}} = (D_\alpha : \alpha < \omega_1)$ be a gap and a collection of dense sets of $\mathbb{P}_{\mathbf{A}}$ in $V[G_{\omega_2}]$ respectively. Then, there is $\lambda < \omega_2$ such that both \mathbf{A} and $\vec{\mathcal{D}}$ are in $V[G_\lambda]$. Since \mathbf{A} is a gap in $V[G_{\omega_2}]$ then is a gap in $V[G_\lambda]$ and there is $\xi \geq \lambda$ such that $\pi(\xi) = (\lambda, \gamma)$ and the γ^{th} name in $V^{\mathbb{P}_\lambda}$ is a name for \mathbf{A} . It follows that there is a $\vec{\mathcal{D}}$ -generic filter in $V[G_{\xi+1}] \subset V[G_{\omega_2}]$ and the proof is finished. \square

This applied to a gap $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ forces $\Gamma \subset \omega_1$ uncountable without the property in Definition 1.2. This shows that there are no T-gaps. Thus, the proof is finished once we show the following.

Claim 5.4. Forcing with $\mathbb{P}_{\mathbf{A}}$ preserves S-gaps.

Proof. Suppose that one $\mathbb{P}_{\mathbf{A}}$ kills an S-gap $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$.

Then $\mathbb{P}_{\mathbf{A}}$ forces $\dot{\Gamma} \subset \omega_1$ uncountable without property (3) of Proposition 2.8 i.e, for every $\alpha < \beta$

$$\mathbb{P}_{\mathbf{A}} \Vdash \alpha, \beta \in \dot{\Gamma} \Rightarrow (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \emptyset$$

Since $\dot{\Gamma}$ is uncountable we can find (in the ground model) $\Gamma \subset \omega_1$ uncountable and $(p_\alpha : \alpha \in \Gamma) \subset \mathbb{P}_{\mathbf{A}}$ such that

$$p_\alpha \Vdash \alpha \in \dot{\Gamma}$$

In particular, we have

$$\forall \alpha < \beta \in \Gamma \quad \left((a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset \implies p_\alpha \cup p_\beta \notin \mathbb{P}_{\mathbf{A}} \right) \quad (5.2)$$

We may assume that the p_γ 's are disjoint and that they all have some fixed size n and $p_\alpha = (x_{\alpha,i})_{i < n}$ preserves the natural order in \mathbf{A} .

Choose a countable elementary sub-model M of H_{c^+} containing all these objects and let $\gamma = \min(\Gamma \setminus M)$.

Since $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is an S-gap, the elementarity of M gives us the existence of a $\beta \in \Gamma$ above γ such that

$$a_\beta \cap b_\gamma = \emptyset \text{ and } a_\gamma \cap b_\beta = \emptyset \quad (5.3)$$

Choose $k < \omega$ such that

$$a_\gamma \setminus k \subseteq a_\beta \text{ and } b_\gamma \setminus k \subseteq b_\beta \quad (5.4)$$

$$\forall x \in p_\gamma \quad \forall y \in p_\beta \quad y \cap k \not\subseteq x \cap k \quad (5.5)$$

Let $s = a_\beta \cap k$, $t = b_\beta \cap k$ and

$$\Gamma_0 = \{\alpha \in \Gamma : a_\alpha \cap k = s \quad b_\alpha \cap k = t \quad x_{\alpha,i} \cap k = x_{\beta,i} \cap k (i < n)\}$$

Then $\Gamma_0 \in M$ and $\beta \in \Gamma_0 \setminus M$ so Γ_0 is an uncountable subset of Γ . Since $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a S-gap there must exist $\alpha \in \Gamma_0 \cap M$ such that

$$a_\alpha \cap b_\beta = \emptyset \quad a_\beta \cap b_\alpha = \emptyset \quad (5.6)$$

Combining equations (5.3), (5.4) and (5.6) we obtain that

$$a_\alpha \cap b_\gamma = \emptyset \quad a_\gamma \cap b_\alpha = \emptyset \quad (5.7)$$

Form the fact that $\alpha \in \Gamma_0$ and by (5.5) we conclude that

$$\forall x \in p_\gamma \quad \forall y \in p_\alpha \quad y \not\subseteq x \quad (5.8)$$

Thus $p_\alpha \cup p_\gamma \in \mathbb{P}_W$, contradicting (5.2). \square

The previous claim also implies that if \mathbb{P}_α preserves S-gaps, then so does $\mathbb{P}_{\alpha+1}$. Suppose now \mathbb{P}_α preserves S-gaps for every $\alpha < \lambda \leq \omega_2$, with λ limit. If \mathbb{P}_λ kills an S-gap, applying the Δ -system lemma (or a counting argument in case α has countable cofinality) we find $\eta < \lambda$ such that \mathbb{P}_η kills an S-gap. Contradiction, thus \mathbb{P}_λ also preserves S-gaps.

This shows that $V[G_{\omega_2}]$ contains an S-gap, since V does and \mathbb{P}_{ω_2} preserves it, and there are no T-gaps in $V[G_{\omega_2}]$ which finish the proof. \square

Remark 5.5. The method above answers a particular case of Problem 59 of [2]. Particularly, it produces a model of with no Suslin towers but destructible gaps.

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